18.100A PROBLEM SET 7

due May 10th 9:30 am

You can collaborate with other students when working on problems. However, you should write the solutions using your own words and thought.

Problem 1. *Exercise 22.1.1. Page 322.*

Problem 2. *Exercise 22.2.2. Page 322.*

Problem 3. *Exercise 22.2.4. Page 323.*

Problem 4. *Exercise 22.3.3. Page 323.*

Problem 5. *Exercise 22.6.5.(a) Page 324. (Hint: Integration.)*

Definition (Stereographic projection)

We define a map $P : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$P(x,y) = \Big(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{-1+x^2+y^2}{1+x^2+y^2}\Big).$$

Then, the image $P(\mathbb{R}^2)$ is $S^2\backslash\{(0,0,1)\}$ where S^2 is the unit sphere $\{(v_1,v_2,v_3) \in \mathbb{R}^3 : v_1^2+v_2^2+v_3^2 = 1\}$. Moreover, P is one-to-one.

Namely, P is mapping a plane to a sphere. So, if Kyeongsu considers the earth flat while Archimedes considers the earth round, the map P provides the relation between points of the earth in their own minds.

Problem 6 (20 points)**.** *We define a function $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by*

$$d\big((x_1,y_1),(x_2,y_2)\big) = \sqrt{1 - P(x_1,y_1)\cdot P(x_2,y_2)},$$

where $P(x_1,y_1)\cdot P(x_2,y_2)$ implies the dot product of the three dimensional unit vectors $P(x_1,y_1), P(x_2,y_2)$.

Show that the following hold

(1) $d \geq 0$, *and* $d\big((x_1,y_1),(x_2,y_2)\big) = 0$ *if and only if* $(x_1,y_1) = (x_2,y_2)$.

(2) $d\big((x_1,y_1),(x_2,y_2)\big) = d\big((x_2,y_2),(x_1,y_1)\big)$.

(3) $d\big((x_1,y_1),(x_2,y_2)\big) + d\big((x_2,y_2),(x_3,y_3)\big) \geq d\big((x_1,y_1),(x_3,y_3)\big)$.

Namely, d is a distance of $\mathbb{R}^2$. You can use the facts given in the blue lines.

Remark. We define a function $\theta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$\theta\big((x_1,y_1),(x_2,y_2)\big) = \arccos|P(x_1,y_1)\cdot P(x_2,y_2)|.$$

Namely, θ is the angle between the two unit vectors $P(x_1,y_1), P(x_2,y_2)$. Then, θ implies the length of the shortest curve on the unit sphere, which connects the two points $P(x_1,y_1), P(x_2,y_2)$.

One can prove that the angle function θ is a distance by using the Pythagoras theorem and elementary geometry as like ancient Greek mathematicians.

Welcome to Geometry!